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Sketching of functions as a digital task with automated feedback

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Outline of this talk

Introduction

Inner design of the task

Reflecting from a didactic point of view

Outline

Technical presentation and discussion of our new digital task

1. Motivation and a presentation of the task
2. A look behind the scenes:
   How does our task work?
3. A discussion from a didactic point of view:
   details, potentials and limits

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Transitions between representations

We want students to deliberately use these and switch between them.

from \ to	table	graph	term
table			
graph			
term			

(Leuders & Prediger, 2005)

Transitions between representations regarding STACK

What does STACK support?

from \ to	table	graph	term
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Transitions between representations regarding STACK

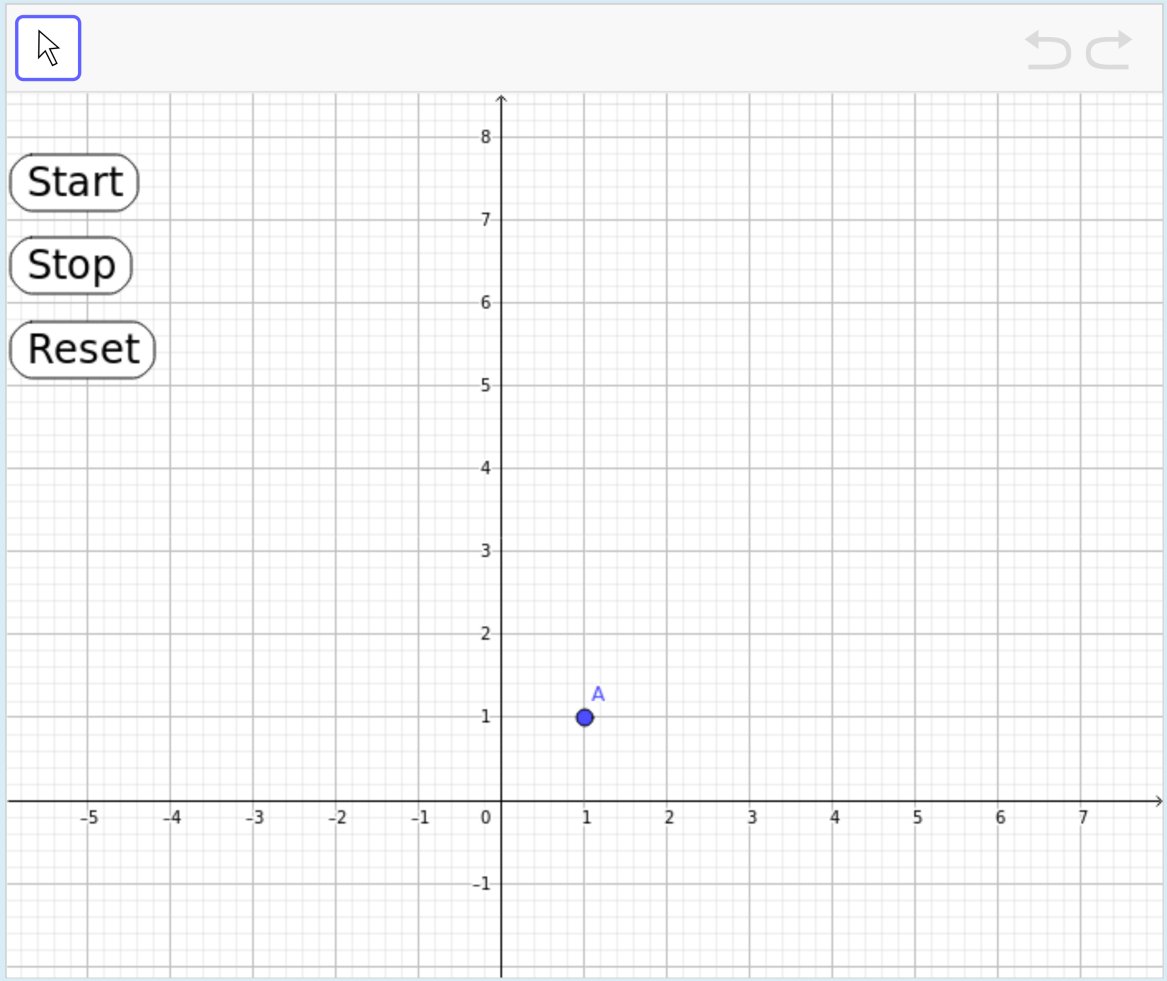
What does STACK support?

from \ to	table	graph	term
table	✔	✘	✔
graph	✔	✘	✔
term	✔	✘	✔

The task

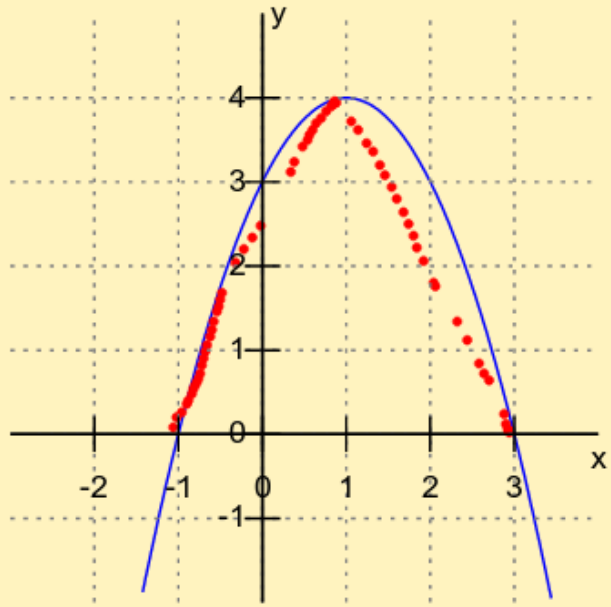
Consider the function $f(x) = -x^2 + 2x + 3$.

Move the point A and use its trace to sketch the function's graph over the interval $-1 \leq x \leq 3$.



Feedback example - imprecise sketch

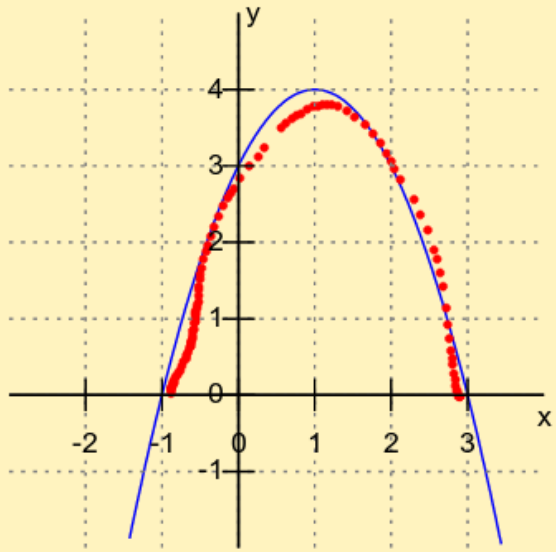
The following coordinate system displays the graph of the function and your sketched trace in the form of its descrete points:



Your sketched trace does not suit the graph of the function or it is not accurate enough.

Feedback example - better sketch with more feedback

The following coordinate system displays the graph of the function and your sketched trace in the form of its descrete points:



Your sketched trace displays some inaccuracies in comparison to the function's graph.

You should keep an eye on important points like x-intercepts or extrema. Try to depict those more precisely next time.

You did not depict the maximum at $P(1|4)$ accurately enough.

2nd Paradigm change?

- First paradigm change: from MC/SC to free algebra input

  strategy of only validating the given answer-options is no longer an option (C. J. Sangwin & Jones, 2017)

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- Now a second paradigm change?

  Means of input become (or do appear) analog instead of digital

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Software framework

(C. Sangwin, 2013; Vasko, 2018)

Software framework

- GeoGebra (DGS)

- STACK

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Software framework

- GeoGebra (DGS)
- JavaScript (linking both technologies)
- STACK

(C. Sangwin, 2013; Vasko, 2018)

Initial idea

We wanted to utilize a (discrete) trace of points by dragging a point in a dynamic geometry software.

- This makes drawing and/or sketching possible
- Enough points make a sketch look continuous (enough)
- A list of points $\Rightarrow$ discrete and limited number of arguments as STACK input
- STACK can handle and evaluate such lists and generate qualitative feedback

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User experience

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User experience $\leftrightarrow$ Back end

PRT-Feedback

Now we want to

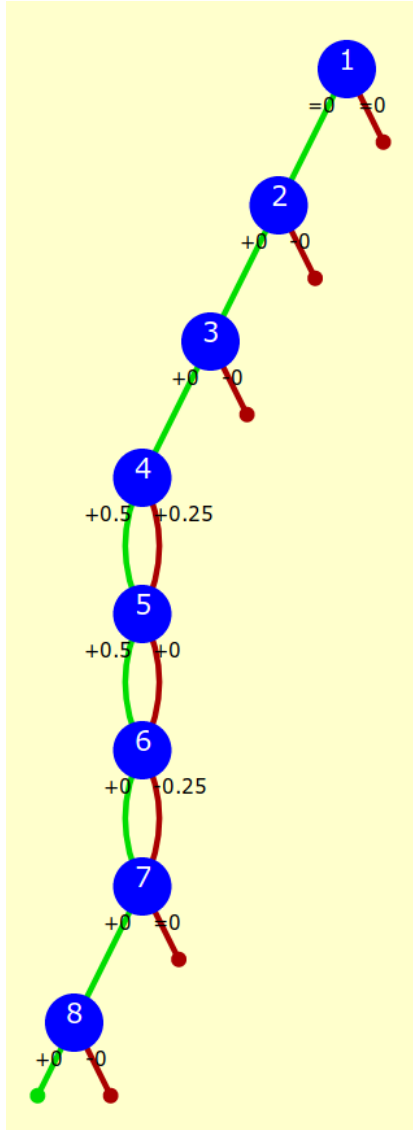
- explain the feedback tree in detail
- explicate thoughts and design decisions we made

Disclaimer

Our task is a prototype in ongoing development.

Thresholds etc. are subject to change.

Abstract tree structure



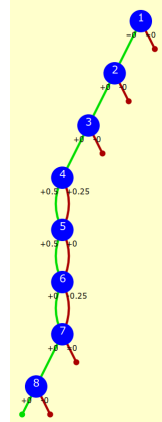
Basic Feedback

3 hard criteria: A negative exit at the first 3 nodes ends the evaluation process

PRT node 1: Check the given amount of points

Thoughts

- A complete sketch should roughly cover the area of the graph
- A substantial analysis of the trace requires a minimal amount of points
- How many points should be required?
- Different input devices do have different levels of precision



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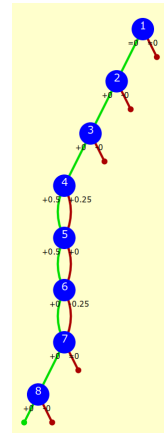
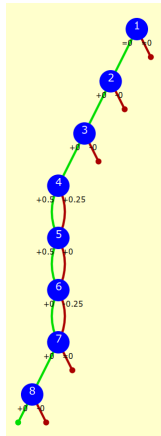
Design decision: A minimum of 30 points is required



PRT node 2: Check the given amount of points in intervals

Thoughts

- A sketch should cover all parts of the graph, i. e. no (huge) ’plot holes’
- A student could try to draw a sufficient number of points in an ’easy’ segment



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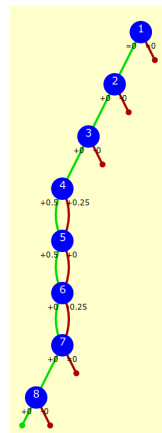
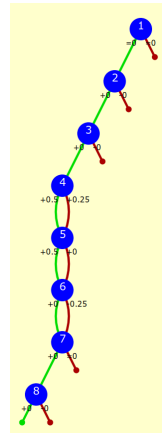
Design decision: 4 intervals and 3 points per interval are required



PRT node 3: Is the sketch on average 'somewhat' accurate?

Calculate the distance to the graph of each point drawn; 2 approaches:

- Horizontal distance (y-direction)
  - Pro: Easily calculated
  - Con: Inaccurate at high slopes



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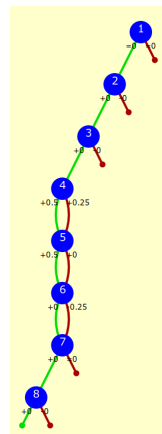
  - Pro: Accurate at high slopes
  - Con: Slower computable
  - Con: Assumption of "minimal error"



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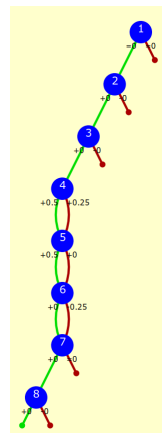
Design decision: Use minimal distance, but with relatively small thresholds ($\varnothing > 0.12$)



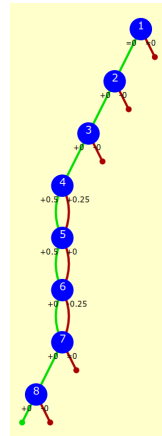
Elaborated Feedback

Soft criteria: All remaining nodes create more detailed feedback for the students

PRT node 4: Is the sketch on average very accurate?

Thoughts

- A sketched plot might be okay but not good
- Accuracy should be considered

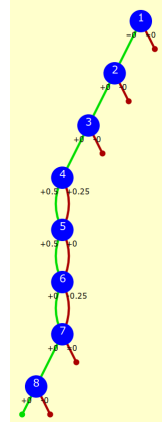


PRT node 4: Is the sketch on average very accurate?

Thoughts

- A sketched plot might be okay but not good
- Accuracy should be considered

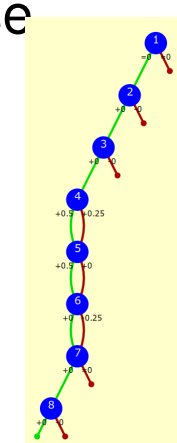
Design decision: Use minimal distance, but with even smaller thresholds ($\varnothing > 0.09$)



PRT node 5: A closer look at the maximum and the zeros

$f(x) = x^2 + 2x + 3$ implicates a maximum at $x = 1$ and zeros at $x = -1$ and $x = 3$.

- Maximum and zeros are very important properties of f
- However, drawing accuracy might be limited (especially with a mouse)
- It should suffice that sketched points in an area around those special spots are close (and do exist)



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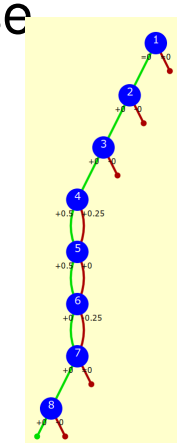
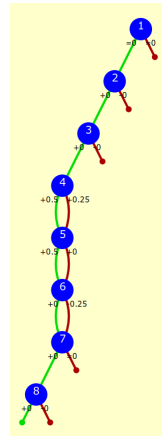
Design decision: Check sketched points with a x-wise distance < 0.2 regarding their minimal distance to the special spots.



PRT node 6: Were the interval limits respected?

- The task gives specific interval limits
- On the other hand: some inaccuracies are okay



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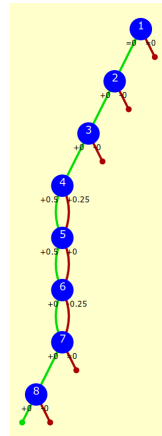
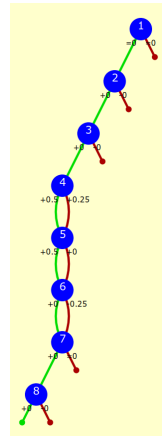
Design decision: Up to 10 values outside $[-1; 3]$ are okay.



PRT node 7: Check for (vastly) runaway points

Thoughts

- An accurate sketch should not have points which deviate vastly
- On the other hand, minor mistakes might occur (unintentionally?)



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Thoughts

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- Or
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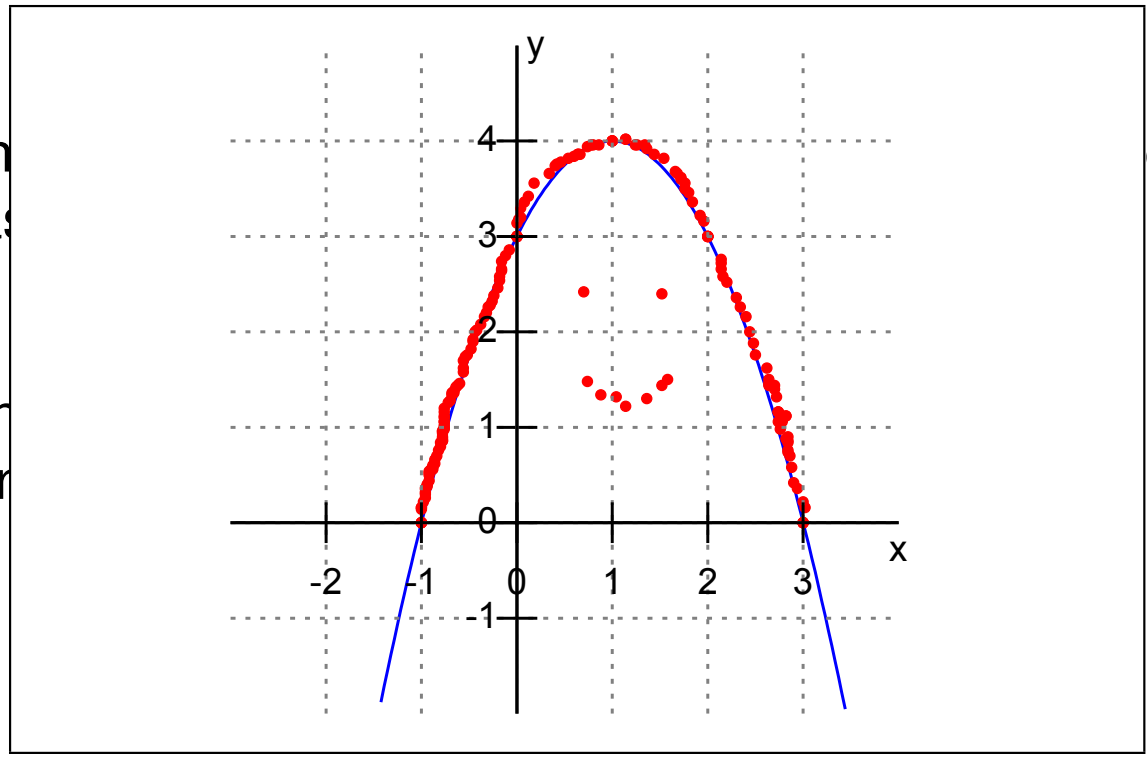

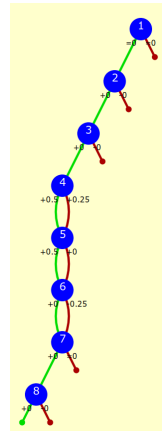

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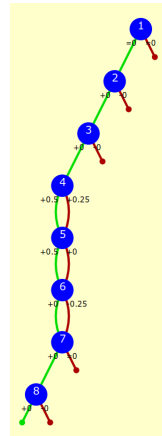

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- An accurate sketch should not have points which deviate vastly
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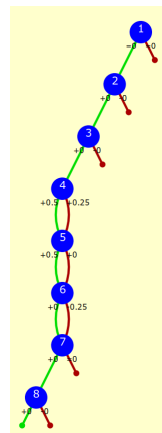
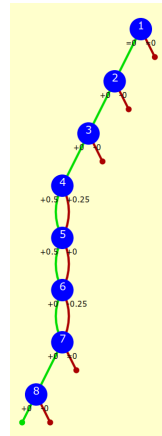
Design decision: Up to 10 points with a distance > 0.3 are tolerable.



PRT node 8: Check for closer runaway points

Thoughts

- A sketch might be thoroughly drawn, but some minor inaccuracies might occur
- This is detailed criticism on a high level!
- Still, feedback might rise the learners awareness



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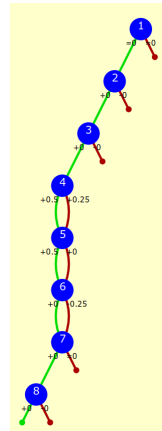
Design decision: 10 values with a distance > 0.2 trigger this feedback.
Examples for those points are explicated for the student.



Unattended polishing

- The applet does not safe the trace and shows up empty after submitting
- Missing labels of the both axis
- Unintuitive applet-interface
- User experienced precision in own sketches
- ...

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Reflecting from a didactic point of view

Discrete or continuous?

Is the point-wise sketching adequate? Or would be continuous more pen-like interface be superior?

Discrete or continuous?

Is the point-wise sketching adequate? Or would be continuous more pen-like interface be superior?

Better question: What are qualitative (and maybe subtle) differences between those two approaches?

Didactic transposition process

(Winsløw, 2011)

- What is the meta-knowledge (about functions) this task does refer to?
  - Abstract definition of a function?
  - What is a function, what a representation?

Dominant representations systems regarding functions

$f : A \to B$
$x \mapsto f(x)$
3 parts: Domain, codomain and mapping.

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At school functions are usually given as (Leuders & Prediger, 2005)

- a term
- a graph
- a table
- other forms (e. g. narratively in a context)

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  - Intelligent sketching (maximum and zeros)?
  - Systematic approaches (calculating enough points of the graph)
  - Drill practice or deliberate practice? (Lehtinen et al., 2017)

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  - Drill practice or deliberate practice? (Lehtinen et al., 2017)
- What do we actually teach?
  - What role would such a task play in a learning environment?
- What does a student actually learn? (e. g. Even, 1998)
  - In an **ideal world** students learn exactly what we want to teach
  - But students might learn to use online function plotters instead
  - Or isolated strategies for a limited set of tasks
  - ...

Thinking with and about functions

1. Functions as a mapping (dependence of a magnitude from another)
2. Covariation of magnitudes
3. Functions as a whole

(Vollrath & Weigand, 2007)

What role do these aspects play explicitly or implicitly?

Possible Misconceptions

- First quadrant is always most important
- A plotted graph does always show all interesting parts (i. e. minimums, maximums, zeros, ...)
- A function should always be plotted
- Every function can be plotted 'nicely'
- ...

Didactic reduction

One task alone cannot get everything right

Interesting perspectives (which are possible as of today)

- All transitions between representations (graph, term, table) are now realizable in digital tasks
- We can make students enter a function and let them sketch it
- Quick-sketching: sketching a graph under time-pressure
- Sketching derivatives or integrals of functions
- Sketching areas, e. g. sets

Limits

- Technical issues (time to create etc.) can limit creativity here
- 2D is possible, 3D might be difficult
- Detecting anti-examples of graphs
- Automated qualitative statements about sketches are difficult to make
  (e. g. 'hey, you got the correct form of the graph but it's translated to the right by one unit')

Following steps

- Interface optimizations
- Task variations and randomization
- Planned Bachelor's thesis: Field testing and scientific evaluation

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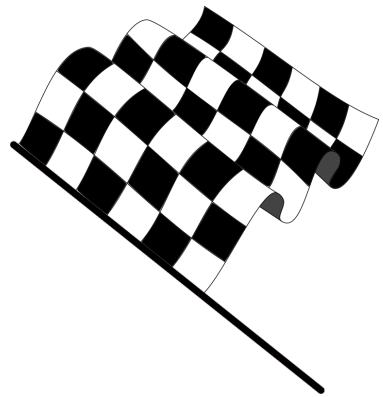
Thank you for your attention!



Now we are looking forward to answering your questions and a discussion with you.